# 9
# Combinations

**COMBINATIONS** *Each of the different groups or selections that can be made out of a given number of things by taking some or all of them at a time, irrespective of their arrangements, is called a combination.*

## ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *All combinations of three persons A, B, C taken two at a time are*

*AB, AC, BC.*

Clearly, *AB* and *BA* represent the same combination but they represent different permutations.

EXAMPLE 2 *All combinations of four persons A, B, C, D taken two at a time are*

*AB, AC, AD, BC, BD, CD.*

EXAMPLE 3 *All combinations of four persons A, B, C, D taken three at a time are*

*ABC, ABD, ACD, BCD.*

**DIFFERENCE BETWEEN A PERMUTATION AND A COMBINATION**

In a combination, only a group is made and the order in which the objects are arranged is immaterial.

On the other hand, in a permutation, not only the group is formed, but also an arrangement in a definite order is considered.

REMARK In general, we use the word *arrangements* for permutations and the word *selections* for combinations.

EXAMPLE 4 (i) *AB* and *BA* are two different permutations but each represents the same combination.

(ii) *ABC, ACB, BAC, BCA, CAB, CBA* are six different permutations but each one represents the same combination, namely *ABC*.

**NOTATION** *The number of all combinations of n things, taken r at a time, is denoted by* ${}^{n}C_{r}$*, or* $C(n, r)$*, where n and r are integers such that* $n > 0, r \geq 0$ *and* $n \geq r$.

### Combinations of *n* Different Objects

THEOREM 1 *The number of all combinations of n distinct objects, taken r at a time, is given by*

$${}^{n}C_{r} = \frac{n!}{(r!) \times (n-r)!}.$$

PROOF Let the number of all combinations of $n$ objects, taken $r$ at a time, be $x$.

Then, ${}^nC_r = x$.

Now, each combination contains $r$ objects, which may be arranged among themselves in $r\,!$ ways.

Thus, each combination gives rise to $r\,!$ permutations.

$\therefore$ $x$ combinations will give rise to $x \times (r\,!)$ permutations.

So, the number of permutations of $n$ things, taken $r$ at a time, is $x \times (r\,!)$.

Consequently, ${}^nP_r = x \times (r\,!) = {}^nC_r \times (r\,!)$.

$$\therefore \quad {}^nC_r = \frac{{}^nP_r}{r\,!} = \frac{n\,!}{(r\,!) \times (n-r)\,!} \quad \left[\because\ {}^nP_r = \frac{n\,!}{(n-r)\,!}\right].$$

**Corollary 1** *Prove that* ${}^nC_r = \dfrac{n(n-1)(n-2)\ldots \text{to } r \text{ factors}}{r\,!}$.

PROOF We have

$$\begin{aligned} {}^nC_r &= \frac{n\,!}{(r\,!) \times (n-r)\,!} \\ &= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1}{(r\,!) \times [(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1]} \\ &= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r\,!} \\ &= \frac{n(n-1)(n-2)\ldots \text{to } r \text{ factors}}{r\,!}. \end{aligned}$$

Hence, ${}^nC_r = \dfrac{n(n-1)(n-2)\ldots \text{to } r \text{ factors}}{r\,!}$.

EXAMPLE 1 *Evaluate* ${}^{11}C_4$.

SOLUTION $${}^{11}C_4 = \frac{11 \times 10 \times 9 \times 8}{4\,!} = \frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = 330.$$

**Corollary 2** *Prove that* ${}^nC_n = {}^nC_0 = 1$.

PROOF We have

$${}^nC_r = \frac{n\,!}{(r\,!) \times (n-r)\,!}. \qquad \ldots \text{(i)}$$

Putting $r = n$ in (i), we get

$${}^nC_r = \frac{n\,!}{(n\,!) \times (n-n)\,!} = \frac{1}{0\,!} = 1 \qquad [\because\ 0\,! = 1].$$

Putting $r = 0$ in (i), we get

$${}^nC_0 = \frac{n\,!}{(0\,!) \times (n-0)\,!} = \frac{n\,!}{1 \times (n\,!)} = 1 \qquad [\because\ 0\,! = 1].$$

Hence, ${}^nC_n = {}^nC_0 = 1$.

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**Corollary 3** *Prove that* $^nC_r = \dfrac{^nP_r}{r!}$.

PROOF We have

$$^nC_r = \frac{n!}{(r!)\times(n-r)!} = \frac{1}{(r!)}\cdot\left\{\frac{n!}{(n-r)!}\right\} = \frac{^nP_r}{r!}.$$

EXAMPLE 2 *If* $^nP_r = 720$ *and* $^nC_r = 120$, *find* $r$.

SOLUTION We have

$$^nC_r = \frac{^nP_r}{r!} \Rightarrow \frac{720}{r!} = 120 \Rightarrow r! = \frac{720}{120} = 6 = (3\times2\times1) = 3!$$
$$\Rightarrow r = 3.$$

Hence, $r = 3$.

**SUMMARY OF THE ABOVE RESULTS**

(i) $^nC_r = \dfrac{n!}{(r!)\times(n-r)!} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$

(ii) $^nC_r = \dfrac{n(n-1)(n-2)\ldots \text{to } r \text{ factors}}{r!}$

(iii) $^nC_n = {^nC_0} = 1$

(iv) $^nC_r = \dfrac{^nP_r}{r!}$

## PROPERTIES OF $^nC_r$

THEOREM 1 *For* $0 \le r \le n$, *prove that* $^nC_r = {^nC_{n-r}}$.

PROOF We have

$$^nC_{n-r} = \frac{n!}{(n-r)!\times\{n-(n-r)\}!} = \frac{n!}{(n-r)!\times(r!)} = {^nC_r}.$$

Hence, $^nC_r = {^nC_{n-r}}$.

EXAMPLE 1 *Evaluate* $^{100}C_{98}$.

SOLUTION Using the result $^nC_r = {^nC_{n-r}}$, we get

$$^{100}C_{98} = {^{100}C_{(100-98)}} = {^{100}C_2} = \frac{100\times99}{2} = 4950.$$

THEOREM 2 *If* $^nC_x = {^nC_y}$ *and* $x \ne y$ *then prove that* $x + y = n$.

PROOF We have

$$^nC_x = {^nC_y} = {^nC_{n-y}} \qquad [\because\ {^nC_r} = {^nC_{n-r}}]$$
$$\Rightarrow\ x = y \text{ or } x = n - y$$
$$\Rightarrow\ x = n - y \qquad [\because\ x \ne y \text{ (given)}]$$
$$\Rightarrow\ x + y = n.$$

Hence, $^nC_x = {^nC_y}$ and $x \ne y$ then $x + y = n$.

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**EXAMPLE 2** *(i) If $^nC_{18} = {^nC_{12}}$ then find the value of $^{32}C_n$.*

*(ii) If $^{10}C_r = {^{10}C_{r+4}}$ then find the value of $^5C_r$.*

**SOLUTION** (i) $^nC_{18} = {^nC_{12}} \Rightarrow n = (18+12) = 30.$

$$\therefore \quad {^{32}C_n} = {^{32}C_{30}} = {^{32}C_{32-30}} = {^{32}C_2} = \frac{32 \times 31}{2\,!} = 496.$$

(ii) $^{10}C_r = {^{10}C_{r+4}} \Rightarrow r + (r+4) = 10 \Rightarrow r = 3.$

$$\therefore \quad {^5C_r} = {^5C_3} = {^5C_{5-3}} = {^5C_2} = \frac{5 \times 4}{2\,!} = 10.$$

**THEOREM 3** *Let $1 \le r \le n$. Then, prove that*

$$^nC_r + {^nC_{r-1}} = {^{n+1}C_r}. \quad [Pascal's\ rule]$$

**PROOF** We have

$$\begin{aligned}
&{^nC_r} + {^nC_{r-1}} \\
&= \frac{n!}{r!\cdot(n-r)!} + \frac{n!}{(r-1)!\cdot\{n-(r-1)\}!} \\
&= \frac{n!}{r!\cdot(n-r)!} + \frac{n!}{(r-1)!\cdot(n-r+1)!} \\
&= \frac{(n!)\cdot(n-r+1)}{r!\cdot(n-r+1)!} + \frac{n!\cdot r}{r!\cdot(n-r+1)!} \\
&= \left\{\frac{n!}{r!\cdot(n-r+1)!}\right\}\cdot\{n-r+1+r\} \\
&= \frac{(n+1)\cdot(n!)}{(r!)\cdot(n-r+1)!} \\
&= \frac{(n+1)!}{(r!)(n+1-r)!} \\
&= {^{n+1}C_r}.
\end{aligned}$$

Hence, $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$.

**THEOREM 4** *If $1 \le r \le n$, prove that $n \times {^{n-1}C_{r-1}} = (n-r+1) \times {^nC_{r-1}}$.*

**PROOF** We have

$$\begin{aligned}
n \times {^{n-1}C_{r-1}} &= n \times \frac{(n-1)!}{(r-1)! \times [(n-1)-(r-1)]!} \\
&= \frac{n \times (n-1)!}{(r-1)! \times (n-r)!} \\
&= \frac{(n!) \times (n-r+1)}{(r-1)! \times (n-r)! \times (n-r+1)}
\end{aligned}$$

[multiplying num. and denom. by $(n-r+1)$]

$$\begin{aligned}
&= (n-r+1) \times \frac{n!}{(r-1)! \times (n-r+1)!} \\
&= (n-r+1) \times {^nC_{r-1}}.
\end{aligned}$$

Hence, $n \times {^{n-1}C_{r-1}} = (n-r+1) \times {^nC_{r-1}}$.

REMARK The above result may be written as

$$\frac{{}^nC_{r-1}}{{}^{n-1}C_{r-1}} = \frac{n}{(n-r+1)}.$$

THEOREM 5 *If n and r are positive integers such that* $1 \le r \le n$ *then prove that:*

*(i)* $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$ *(ii)* $\frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{n}{r}$

PROOF (i) We know that

$${}^nC_r = \frac{n!}{(r!)\times(n-r)!} \text{ and } {}^nC_{r-1} = \frac{n!}{(r-1)!\times(n-r+1)!}.$$

$$\therefore \quad \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n!}{(r!)\times(n-r)!} \times \frac{(r-1)!\times(n-r+1)!}{n!}$$

$$= \frac{(r-1)!\times(n-r+1)\times(n-r)!}{r\times(r-1)!\times(n-r)!}$$

$$= \frac{n-r+1}{r}.$$

Hence, $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}.$

(ii) We have

$${}^nC_r = \frac{n!}{(r!)\times(n-r)!}$$

and $${}^{n-1}C_{r-1} = \frac{(n-1)!}{(r-1)!\times\{(n-1)-(r-1)\}!} = \frac{(n-1)!}{(r-1)!\times(n-r)!}.$$

$$\therefore \quad \frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{n!}{(r!)\times(n-r)!} \times \frac{(r-1)!\times(n-r)!}{(n-1)!} = \frac{n}{r}.$$

Hence, $\frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{n}{r}.$

**SUMMARY OF THE ABOVE RESULTS**

(i) ${}^nC_r = {}^nC_{n-r}$

(ii) ${}^nC_x = {}^nC_y$ and $x \ne y \Rightarrow x + y = n$

(iii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ [Pascal's rule]

(iv) $\frac{{}^nC_{r-1}}{{}^{n-1}C_{r-1}} = \frac{n}{(n-r+1)}$ (v) $\frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{n}{r}$

(vi) $\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r}$

## SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) ${}^{10}C_3$ (ii) ${}^{11}C_8$ (iii) ${}^{50}C_{48}$ (iv) ${}^{63}C_{63}$

SOLUTION We have

(i) ${}^{10}C_3 = \dfrac{10\times 9\times 8}{3!} = \dfrac{10\times 9\times 8}{3\times 2\times 1} = 120.$

(ii) ${}^{11}C_8 = {}^{11}C_{(11-8)} = {}^{11}C_3 = \dfrac{11\times 10\times 9}{3!} = \dfrac{11\times 10\times 9}{3\times 2\times 1} = 165$

$[\because\ {}^nC_r = {}^nC_{n-r}]$.

(iii) ${}^{50}C_{48} = {}^{50}C_{(50-48)} = {}^{50}C_2 = \dfrac{50\times 49}{2} = 1225.$

(iv) ${}^{63}C_{63} = 1$ $[\because\ {}^nC_n = 1]$.

EXAMPLE 2 *If* ${}^nC_8 = {}^nC_6$, *find* ${}^nC_3$.

SOLUTION We know that ${}^nC_p = {}^nC_q$ and $p \neq q \Rightarrow n = p + q$.

$\therefore\quad {}^nC_8 = {}^nC_6 \Rightarrow n = (8+6) = 14.$

$\therefore\quad {}^nC_3 = {}^{14}C_3 = \dfrac{14\times 13\times 12}{3!} = \dfrac{14\times 13\times 12}{3\times 2\times 1} = 364.$

EXAMPLE 3 *If* ${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 3:4:5$, *find the values of n and r.*

SOLUTION We have, $\dfrac{{}^nC_r}{{}^nC_{r+1}} = \dfrac{r+1}{n-r}$. ... (i)

Putting $\dfrac{{}^nC_r}{{}^nC_{r+1}} = \dfrac{4}{5}$ in (i), we get

$\dfrac{r+1}{n-r} = \dfrac{4}{5} \Rightarrow 5r + 5 = 4n - 4r \Rightarrow 4n - 9r = 5.$ ... (ii)

Replacing $r$ by $(r-1)$ in (i), we get

$\dfrac{{}^nC_{r-1}}{{}^nC_r} = \dfrac{r}{n-r+1} \Rightarrow \dfrac{r}{n-r+1} = \dfrac{3}{4}$

$\Rightarrow 3n - 3r + 3 = 4r \Rightarrow 3n - 7r = -3.$ ... (iii)

On solving (ii) and (iii), we get $n = 62$ and $r = 27$.

EXAMPLE 4 *If* ${}^{2n}C_3 : {}^nC_3 = 11:1$, *find n.*

SOLUTION We have

$${}^{2n}C_3 = \frac{2n(2n-1)(2n-2)}{3!} = \frac{2n(n-1)(2n-1)}{3}$$

and $${}^nC_3 = \frac{n(n-1)(n-2)}{3!} = \frac{n(n-1)(n-2)}{6}.$$

$$\therefore\quad \frac{{}^{2n}C_3}{{}^nC_3} = \frac{2n(n-1)(2n-1)}{3} \times \frac{6}{n(n-1)(n-2)} = \frac{4(2n-1)}{(n-2)}.$$

Now, $\dfrac{{}^{2n}C_3}{{}^nC_3} = \dfrac{11}{1} \Rightarrow \dfrac{4(2n-1)}{(n-2)} = \dfrac{11}{1}$

$$\Rightarrow\ 8n-4=11n-22\ \Rightarrow\ 3n=18\ \Rightarrow\ n=6.$$

Hence, $n=6$.

EXAMPLE 5 *Prove that the product of r consecutive positive integers is divisible by r !.*

SOLUTION Let the $r$ consecutive positive integers be $(n+1), (n+2), \ldots, (n+r)$.

Then, their product $= (n+1)(n+2)(n+3)\ldots(n+r)$

$$= \frac{(n!)(n+1)(n+2)(n+3)\ldots(n+r)}{n!}$$

$$= \frac{(n+r)!}{n!} = (r!)\cdot\frac{(n+r)!}{(r!)\times[(n+r)-r]!}$$

$= (r!)\times {}^{n+r}C_r$, which is divisible by $r!$, since ${}^{n+r}C_r$ is a positive integer.

EXAMPLE 6 *If* ${}^nP_r = {}^nP_{r+1}$ *and* ${}^nC_r = {}^nC_{r-1}$, *find n and r.*

SOLUTION ${}^nP_r = {}^nP_{r+1} \Rightarrow \dfrac{n!}{(n-r)!} = \dfrac{n!}{(n-r-1)!}$

$\Rightarrow (n-r)! = (n-r-1)!$

$\Rightarrow (n-r)\times(n-r-1)! = (n-r-1)!$

$\Rightarrow n-r=1.$ … (i)

${}^nC_r = {}^nC_{r-1} \Rightarrow \dfrac{{}^nC_r}{{}^nC_{r-1}} = 1$

$\Rightarrow \dfrac{n-r+1}{r} = 1 \quad \left[\because \dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}\right]$

$\Rightarrow n-2r=-1.$ … (ii)

Solving (i) and (ii), we get $n=3$ and $r=2$.

EXAMPLE 7 *Prove that* ${}^{2n}C_n = \dfrac{2^n\times[1\cdot3\cdot5\ldots(2n-1)]}{n!}$.

SOLUTION ${}^{2n}C_n = \dfrac{(2n)!}{(n!)(2n-n)!} = \dfrac{(2n)!}{(n!)^2}$

$$= \frac{(2n)(2n-1)(2n-2)\ldots4\cdot3\cdot2\cdot1}{(n!)^2}$$

$$= \frac{[(2n)(2n-2)(2n-4)\ldots4\cdot2]\times[(2n-1)(2n-3)\ldots5\cdot3\cdot1]}{(n!)^2}$$

$$= \frac{2^n[n(n-1)(n-2)\ldots2\cdot1]\times[(2n-1)(2n-3)\ldots5\cdot3\cdot1]}{(n!)^2}$$

$$= \frac{2^n\times(n!)\times[1\cdot3\cdot5\ldots(2n-3)(2n-1)]}{(n!)^2}$$

$$= \frac{2^n\times[1\cdot3\cdot5\ldots(2n-3)(2n-1)]}{(n!)}.$$

Hence, ${}^{2n}C_n = \dfrac{2^n \times \{1 \times 3 \times 5 \times \ldots \times (2n-1)\}}{n!}$.

EXAMPLE 8 *Prove that*

$${}^{4n}C_{2n} : {}^{2n}C_n = \{1 \times 3 \times 5 \times \ldots \times (4n-1)\} : \{1 \times 3 \times 5 \times \ldots \times (2n-1)\}^2.$$

SOLUTION We have

$$\frac{{}^{4n}C_{2n}}{{}^{2n}C_n} = \frac{(4n)!}{(2n)! \times (2n)!} \times \frac{(n!) \times (n!)}{(2n)!}$$

$$= \frac{(4n)! \times (n!)^2}{(2n)! \times [(2n)!]^2}$$

$$= \frac{[1 \cdot 2 \cdot 3 \cdot 4 \ldots (4n-1)(4n)] \times (n!)^2}{(2n)! \times [1 \cdot 2 \cdot 3 \cdot 4 \ldots (2n-2)(2n-1)(2n)]^2}$$

$$= \frac{[1 \cdot 3 \cdot 5 \ldots (4n-1)] \times [2 \cdot 4 \cdot 6 \ldots (4n)] \times (n!)^2}{(2n)! \times [1 \cdot 3 \cdot 5 \ldots (2n-1)]^2 \times [2 \cdot 4 \cdot 6 \ldots (2n-2)(2n)]^2}$$

$$= \frac{[1 \cdot 3 \cdot 5 \ldots (4n-1)] \times 2^{2n} \times [1 \cdot 2 \cdot 3 \ldots (2n)] \times (n!)^2}{(2n)! \times [1 \cdot 3 \cdot 5 \ldots (2n-1)]^2 \times 2^{2n} \times (n!)^2}$$

$$= \frac{[1 \cdot 3 \cdot 5 \ldots (4n-1)]}{[1 \cdot 3 \cdot 5 \ldots (2n-1)]^2}.$$

$\therefore \quad {}^{4n}C_{2n} : {}^{2n}C_n = [1 \cdot 3 \cdot 5 \ldots (4n-1)\} : \{1 \cdot 3 \cdot 5 \ldots (2n-1)\}^2.$

EXAMPLE 9 *If* ${}^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$, *find n.*

SOLUTION We have

$${}^{n+2}C_8 = \frac{(n+2)!}{(8!)(n+2-8)!} = \frac{(n+2)!}{(8!)(n-6)!} \qquad \left[\because\ {}^nC_r = \frac{n!}{(r!) \cdot (n-r)!}\right]$$

and $${}^{n-2}P_4 = \frac{(n-2)!}{(n-2-4)!} = \frac{(n-2)!}{(n-6)!} \qquad \left[\because\ {}^nP_r = \frac{n!}{(n-r)!}\right].$$

Now, $\dfrac{{}^{n+2}C_8}{{}^{n-2}P_4} = \dfrac{57}{16}$

$$\Rightarrow \quad \frac{(n+2)!}{(8!)(n-6)!} \times \frac{(n-6)!}{(n-2)!} = \frac{57}{16}$$

$$\Rightarrow \quad \frac{(n+2)(n+1)n(n-1) \cdot (n-2)!}{(n-2)!} = \left(\frac{57}{16} \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1\right)$$

$\Rightarrow \quad (n+2)(n+1)n(n-1) = 21 \times 20 \times 19 \times 18$

$\Rightarrow \quad n-1 = 18 \ \Rightarrow\ n = 19.$

Hence, $n = 19$.

## EXERCISE 9A

1. Evaluate:
   (i) ${}^{20}C_4$ (ii) ${}^{16}C_{13}$ (iii) ${}^{90}C_{88}$
   (iv) ${}^{71}C_{71}$ (v) ${}^{n+1}C_n$ (vi) $\sum_{r=1}^{6} {}^6C_r$
2. Verify that:
   (i) ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7 = 0$ (ii) ${}^{10}C_4 + {}^{10}C_3 = {}^{11}C_4$
3. (i) If ${}^nC_7 = {}^nC_5$, find *n*.
   (ii) If ${}^nC_{14} = {}^nC_{16}$, find ${}^nC_{28}$.
   (iii) If ${}^nC_{16} = {}^nC_{14}$, find ${}^nC_{27}$.
4. (i) If ${}^{20}C_r = {}^{20}C_{r+6}$, find *r*.
   (ii) If ${}^{18}C_r = {}^{18}C_{r+2}$, find ${}^rC_5$.
5. If ${}^nC_{r-1} = {}^nC_{3r}$, find *r*.
6. If ${}^{2n}C_3 : {}^nC_3 = 12:1$, find *n*.
7. If ${}^{15}C_r : {}^{15}C_{r-1} = 11:5$, find *r*.
8. If ${}^nP_r = 840$ and ${}^nC_r = 35$, find the value of *r*.
9. If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, find *n* and *r*.
10. If ${}^{n+1}C_{r+1} : {}^nC_r = 11:6$ and ${}^nC_r : {}^{n-1}C_{r-1} = 6:3$, find *n* and *r*.
11. How many different teams of 11 players can be chosen from 15 players?
12. If there are 12 persons in a party and if each two of them shake hands with each other, how many handshakes are possible?
13. How many chords can be drawn through 21 points on a circle?
14. From a class of 25 students, 4 are to be chosen for a competition. In how many ways can this be done?

## ANSWERS (EXERCISE 9A)

**1.** (i) 4845 (ii) 560 (iii) 4005 (iv) 1 (v) $n+1$ (vi) 63
**3.** (i) $n = 12$ (ii) 435 (iii) 4060 **4.** (i) $r = 7$ (ii) 56
**5.** $r = \frac{1}{4}(n+1)$ **6.** $n = 5$ **7.** $r = 5$ **8.** $r = 4$ **9.** $n = 9, r = 3$
**10.** $n = 10, r = 5$ **11.** 1365 **12.** 66 **13.** 210 **14.** 12650

## HINTS TO SOME SELECTED QUESTIONS

**6.** $\frac{2n(2n-1)(2n-2)}{3\times2\times1} \times \frac{3\times2\times1}{n(n-1)(n-2)} = \frac{12}{1}$

$\Rightarrow\ 4(2n-1) = 12(n-2) \Rightarrow 2n-1 = 3n-6 \Rightarrow n = 5.$

**7.** $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} \Rightarrow \frac{{}^{15}C_r}{{}^{15}C_{r-1}} = \frac{15-r+1}{r} = \frac{16-r}{r} \Rightarrow \frac{16-r}{r} = \frac{11}{5}.$

Find *r*.

8. Use $^nC_r = \frac{^nP_r}{r!}$.

9. $\frac{^nC_{r-1}}{^nC_r} = \frac{r}{n-r+1}$ and $\frac{^nC_r}{^nC_{r+1}} = \frac{r+1}{n-r}$.

$\therefore \quad \frac{r}{n-r+1} = \frac{36}{84} = \frac{3}{7} \Rightarrow 3n-3r+3 = 7r \Rightarrow 3n-10r = -3$ ... (i)

and $\frac{r+1}{n-r} = \frac{84}{126} = \frac{2}{3} \Rightarrow 2n-2r = 3r+3 \Rightarrow 2n-5r = 3.$ ... (ii)

10. Use $\frac{^nC_r}{^{n-1}C_{r-1}} = \frac{n}{r}$ and $\frac{^{n+1}C_{r+1}}{^nC_r} = \frac{n+1}{r+1}$.

$\therefore \quad \frac{n}{r} = \frac{6}{3} = \frac{2}{1}$ and $\frac{n+1}{r+1} = \frac{11}{6} \Rightarrow 6n+6 = 11r+11 \Rightarrow 6n-11r = 5.$

Solve $n-2r = 0$ and $6n-11r = 5$.

11. Required number of teams $= {}^{15}C_{11} = {}^{15}C_{(15-11)} = {}^{15}C_4$.

12. Number of handshakes $= {}^{12}C_2$.

13. Number of chords $= {}^{21}C_2$.

14. Required number of ways $= {}^{25}C_4$.

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## PRACTICAL PROBLEMS ON COMBINATIONS

### SOLVED EXAMPLES

EXAMPLE 1 *In how many ways can a cricket eleven be chosen out of a batch of 15 players, if*

*(i) there is no restriction on the selection?*

*(ii) a particular player is always chosen?*

*(iii) a particular player is never chosen?*

SOLUTION (i) Total number of ways of selecting 11 players out of 15

$$= {}^{15}C_{11} = {}^{15}C_{15-11} = {}^{15}C_4 = \frac{15\times14\times13\times12}{4\times3\times2\times1} = 1365.$$

(ii) When a particular player is always chosen, then we will have to choose 10 players out of remaining 14.

$\therefore$ required number of ways

$$= {}^{14}C_{10} = {}^{14}C_{14-10} = {}^{14}C_4 = \frac{14\times13\times12\times11}{4\times3\times2\times1} = 1001.$$

(iii) When a particular player is never chosen, then we will have to choose 11 players out of remaining 14.

$\therefore$ required number of ways

$$= {}^{14}C_{11} = {}^{14}C_{14-11} = {}^{14}C_3 = \frac{14\times13\times12}{3\times2\times1} = 364.$$

EXAMPLE 2 *In how many ways can a committee of 5 members be selected from 6 men and 5 ladies, consisting of 3 men and 2 ladies?*

SOLUTION We have to select (3 men out of 6) and (2 ladies out of 5).

$\therefore$ required number of ways

$$= ({}^6C_3 \times {}^5C_2) = \left(\frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = (20 \times 10) = 200.$$

EXAMPLE 3 *Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can it be formed if at least one woman is to be included?*

SOLUTION The committee can be formed by choosing

(i) 1 woman and 2 men;

or (ii) 2 women and 1 man.

Now, the number of ways of choosing (1 woman out of 2) and (2 men out of 5)

$$= ({}^2C_1 \times {}^5C_2) = \left(2 \times \frac{5 \times 4}{2 \times 1}\right) = (2 \times 10) = 20.$$

And, the number of ways of choosing (2 women out of 2) and (1 man out of 5)

$$= ({}^2C_2 \times {}^5C_1) = (1 \times 5) = 5.$$

Hence, the total number of ways of forming the committee

$$= (20 + 5) = 25.$$

EXAMPLE 4 *A committee of 5 is to be formed out of 6 men and 4 ladies. In how many ways can this be done, when*

*(a) at least 2 ladies are included;*

*(b) at most 2 ladies are included?*

SOLUTION (a) We have to make a selection of

(i) (2 ladies out of 4) and (3 men out of 6)

or (ii) (3 ladies out of 4) and (2 men out of 6)

or (iii) (4 ladies out of 4) and (1 man out of 6).

The number of ways of these selections are:

**Case I** ${}^4C_2 \times {}^6C_3 = 6 \times 20 = 120.$

**Case II** ${}^4C_3 \times {}^6C_2 = 4 \times 15 = 60.$

**Case III** ${}^4C_4 \times {}^6C_1 = 1 \times 6 = 6.$

Hence, the required number of ways $= (120 + 60 + 6) = 186.$

(b) We have to make a selection of

(i) (1 lady out of 4) and (4 men out of 6)

or (ii) (2 ladies out of 4) and (3 men out of 6).

The number of ways of these selections are:

**Case I** ${}^4C_1 \times {}^6C_4 = 4 \times 15 = 60.$

**Case II** ${}^4C_2 \times {}^6C_3 = 6 \times 20 = 120.$

Hence, the required number of ways $= (60 + 120) = 180.$

EXAMPLE 5 *An examination paper containing 12 questions consists of two parts, A and B. Part A contains 7 questions and part B contains 5 questions.*

*A candidate is required to attempt 8 questions, selecting at least 3 from each part. In how many ways can the candidate select the questions?*

SOLUTION Clearly, the candidate may select the questions as under:

(i) (3 out of 7 from *A*) and (5 out of 5 from *B*)

or (ii) (4 out of 7 from *A*) and (4 out of 5 from *B*)

or (iii) (5 out of 7 from *A*) and (3 out of 5 from *B*).

The number of ways of these selections are:

**Case I** $^7C_3 \times {}^5C_5 = 35 \times 1 = 35.$

**Case II** $^7C_4 \times {}^5C_4 = 35 \times 5 = 175.$

**Case III** $^7C_5 \times {}^5C_3 = 21 \times 10 = 210.$

Hence, the required number of ways $= (35 + 175 + 210) = 420.$

EXAMPLE 6 *For the post of 5 teachers, there are 23 applicants. 2 posts are reserved for SC candidates and there are 7 SC candidates among the applicants. In how many ways can the selection be made?*

SOLUTION Clearly, there are 7 SC candidates and 16 non-SC candidates.

So, we have to make a selection of (2 out of 7) and (3 out of 16).

Hence, the number of ways of making the selection

$$= {}^7C_2 \times {}^{16}C_3 = \left(\frac{7 \times 6}{2} \times \frac{16 \times 15 \times 14}{3 \times 2 \times 1}\right) = 11760.$$

EXAMPLE 7 *How many diagonals are there in a polygon of n sides?*

SOLUTION A polygon of $n$ sides has $n$ vertices.

By joining any two of these vertices, we obtain either a side or a diagonal of the polygon.

Number of all straight lines obtained by joining 2 vertices at a time

$= {}^nC_2 = \frac{1}{2}n(n-1).$

These straight lines include $n$ sides of the polygon.

Hence, the number of diagonals of the polygon

$$= \left[\frac{1}{2}n(n-1) - n\right] = \frac{1}{2}n(n-3).$$

EXAMPLE 8 *There are 10 points in a plane, no three of which are in the same straight line, except 4 points, which are collinear.*

*Find (i) the number of lines obtained from the pairs of these points;*

*(ii) the number of triangles that can be formed with vertices as these points.*

SOLUTION (i) Number of lines formed by joining the 10 points, taking 2 at a time

$$= {}^{10}C_2 = \left(\frac{10 \times 9}{2 \times 1}\right) = 45.$$

Number of lines formed by joining the 4 points, taking 2 at a time

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$$= {}^4C_2 = \left(\frac{4 \times 3}{2 \times 1}\right) = 6.$$

But, 4 collinear points, when joined pairwise, give 1 line.

Hence, the required number of straight lines

$$= (45 - 6 + 1) = 40.$$

(ii) Number of triangles formed by 10 points, taking 3 at a time

$$= {}^{10}C_3 = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120.$$

Number of triangles formed by 4 points, taking 3 at a time

$$= {}^4C_3 = {}^4C_{4-3} = {}^4C_1 = 4.$$

But, these 4 points being collinear, so no triangle is formed.

Hence, the required number of triangles $= (120 - 4) = 116$.

EXAMPLE 9 *A box contains 6 red and 5 white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour?*

SOLUTION We may select

(i) (2 red balls out of 6) and (4 white balls out of 5)

or (ii) (3 red balls out of 6) and (3 white balls out of 5)

or (iii) (4 red balls out of 6) and (2 white balls out of 5).

The number of ways of these selections are:

**Case I** $({}^6C_2 \times {}^5C_4) = ({}^6C_2 \times {}^5C_1) = \left(\frac{6 \times 5}{2} \times 5\right) = 75.$

**Case II** $({}^6C_3 \times {}^5C_3) = ({}^6C_3 \times {}^5C_2) = \left(\frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = 200.$

**Case III** $({}^6C_4 \times {}^5C_2) = ({}^6C_2 \times {}^5C_2) = \left(\frac{6 \times 5}{2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = 150.$

Hence, the required number of ways $= (75 + 200 + 150) = 425$.

EXAMPLE 10 *How many committees of 5 persons with a chairperson can be selected from 12 persons?*

SOLUTION Number of ways of selecting a chairperson out of 12 persons

$$= {}^{12}C_1 = 12.$$

Number of ways of selecting 4 persons out of remaining 11

$$= {}^{11}C_4 = \frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = 330.$$

Hence, the required number of ways $= (12 \times 330) = 3960$.

EXAMPLE 11 *A box contains 3 black, 2 white and 4 red balls. In how many ways can 3 balls be drawn from the box, if at least one black ball is to be included in the draw?*

SOLUTION In all, we have 3 black and 6 non-black balls.

We may select:

(i) (1 black ball out of 3) and (2 non-black balls out of 6)

or (ii) (2 black balls out of 3) and (1 non-black ball out of 6)

or (iii) (3 black balls out of 3).

The number of ways of these selections are:

**Case I** $({}^3C_1 \times {}^6C_2) = \left(3 \times \frac{6 \times 5}{2 \times 1}\right) = 45.$

**Case II** $({}^3C_2 \times {}^6C_1) = ({}^3C_1 \times {}^6C_1) = (3 \times 6) = 18.$

**Case III** ${}^3C_3 = 3! = (3 \times 2 \times 1) = 6.$

Hence, the required number of ways $= (45 + 18 + 6) = 69.$

EXAMPLE 12 *A group consists of 7 boys and 4 girls. In how many ways can a team of 5 members be selected, if the team has*

*(i) no girls?*

*(ii) at least 1 boy and 1 girl?*

*(iii) at least 3 girls?*

SOLUTION (i) *When no girl is taken*:

Then, we have to select 5 boys out of 7.

∴ required number of ways

$$= {}^7C_5 = {}^7C_{7-5} = {}^7C_2 = \frac{7 \times 6}{2 \times 1} = 21.$$

(ii) *When at least 1 boy and 1 girl are taken*:

In this case, we may choose:

(1 boy and 4 girls) or (2 boys and 3 girls)

or (3 boys and 2 girls) or (4 boys and 1 girl).

∴ required number of ways

$$= ({}^7C_1 \times {}^4C_4) + ({}^7C_2 \times {}^4C_3) + ({}^7C_3 \times {}^4C_2) + ({}^7C_4 \times {}^4C_1)$$

$$= (7 \times 1) + ({}^7C_2 \times {}^4C_1) + ({}^7C_3 \times {}^4C_2) + ({}^7C_3 \times {}^4C_1)$$

$$= 7 + \left(\frac{7 \times 6}{2 \times 1} \times 4\right) + \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1}\right) + \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times 4\right)$$

$$= (7 + 84 + 210 + 140) = 441.$$

(iii) *When at least 3 girls are taken*:

Then, we may choose:

(3 girls and 2 boys) or (4 girls and 1 boy).

∴ required number of ways

$$= ({}^4C_3 \times {}^7C_2) + ({}^4C_4 \times {}^7C_1)$$

$$= ({}^4C_1 \times {}^7C_2) + (1 \times {}^7C_1) = \left(4 \times \frac{7 \times 6}{2 \times 1}\right) + (1 \times 7)$$

$$= (84 + 7) = 91.$$

EXAMPLE 13 *A bag contains 6 white marbles and 5 red marbles. Find the number of ways in which 4 marbles can be drawn from the bag, if*

*(a) they can be of any colour;*

*(b) 2 must be white and 2 red;*

*(c) they must all be of the same colour.*

SOLUTION (a) *Drawing 4 marbles of any colour*:

Total number of marbles $= (6+5) = 11$.

In this case, we have to draw 4 marbles out of 11.

$\therefore$ required number of ways

$$= {}^{11}C_4 = \frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = 330.$$

(b) *Drawing 2 white and 2 red marbles*:

We may draw:

(2 white marbles out of 6) and (2 red marbles out of 5).

$\therefore$ required number of ways

$$= ({}^6C_2 \times {}^5C_2) = \left(\frac{6 \times 5}{2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = (15 \times 10) = 150.$$

(c) *Drawing 4 marbles of the same colour*:

We may draw:

(4 white marbles out of 6) or (4 red marbles out of 5) [*Note it*]

$\therefore$ required number of ways

$$= ({}^6C_4 + {}^5C_4) = ({}^6C_2 + {}^5C_1) = \left(\frac{6 \times 5}{2 \times 1} + 5\right) = 20.$$

EXAMPLE 14 *We wish to select 6 persons from 8, but if the person A is chosen, then B must be chosen. In how many ways can the selection be made?*

SOLUTION We have two cases: (either *A* is chosen) or (*A* is not chosen).

**Case I** *When A is chosen:*

In this case *A* is chosen and *B* is chosen.

So, we must choose 4 persons out of remaining 6.

$\therefore$ required number of ways $= {}^6C_4 = {}^6C_2 == \frac{6 \times 5}{2 \times 1} = 15.$

**Case II** *When A is not chosen:*

In this case, we have to choose 6 persons out of 7.

$\therefore$ required number of ways $= {}^7C_6 = {}^7C_1 = 7.$

Combining both the cases, we have:

total number of ways $= (15+7) = 22.$

EXAMPLE 15 *Find the number of ways of choosing 4 cards from a pack of 52 playing cards. In how many of these*

*(i) 4 cards are of the same suit?*

*(ii) 4 cards belong to four different suits?*

*(iii) 4 cards are face cards?*

*(iv) 2 are red cards and 2 are black cards?*

*(v) cards are of the same colour?*

SOLUTION Number of ways of choosing 4 cards out of 52

$$= {}^{52}C_4 = \frac{52 \times 51 \times 50 \times 49}{4 \times 3 \times 2 \times 1} = 270725.$$

(i) *Choosing 4 cards of the same suit:*

We have to draw:

(4 cards from 13 cards of diamond)

or (4 cards from 13 cards of spade)

or (4 cards from 13 cards of club)

or (4 cards from 13 cards of heart).

$\therefore$ required number of ways

$$= ({}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4) = 4 \times {}^{13}C_4$$

$$= \frac{4 \times 13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = 2860.$$

(ii) *Choosing 4 cards from 4 different suits:*

We have to draw:

(1 card from 13 cards of diamond)

and (1 card from 13 cards of spade)

and (1 card from 13 cards of club)

and (1 card from 13 cards of heart).

$\therefore$ required number of ways

$$= ({}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1) = (13 \times 13 \times 13 \times 13) = (13)^4.$$

(iii) *Choosing 4 cards out of 12 face cards:*

We have to choose 4 cards out of 12 face cards.

$\therefore$ required number of ways

$$= {}^{12}C_4 = \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} = 495.$$

(iv) *Choosing 2 red cards and 2 black cards:*

We have to choose 2 red cards out of 26 and 2 black cards out of 26.

$\therefore$ required number of ways

$$= ({}^{26}C_2 \times {}^{26}C_2) = \left(\frac{26 \times 25}{2}\right)^2 = (325 \times 325) = 105625.$$

(v) *Choosing all the 4 cards of the same colour:*

We have to choose 4 cards out of 26 black cards or 4 cards out of 26 red cards.

$\therefore$ required number of ways

$$= ({}^{26}C_4 + {}^{26}C_4) = (2 \times {}^{26}C_4) = \left(2 \times \frac{26 \times 25 \times 24 \times 23}{4 \times 3 \times 2 \times 1}\right)$$

$$= 29900.$$

EXAMPLE 16 *From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they be chosen?*

SOLUTION Either 3 particular students join or none of them joins.

**Case I** *When 3 particular students join the party:*

In this case, we have to choose 7 students out of 22.

This can be done in ${}^{22}C_7$ ways.

**Case II** *When 3 particular students do not join the party:*

In this case, we have to choose 10 students out of 22.

This can be done in ${}^{22}C_{10}$ ways.

Hence, the required number of ways $= ({}^{22}C_7 + {}^{22}C_{10}) = 817190$.

EXAMPLE 17 *A boy has 3 library tickets and 8 books of his interest in the library. Of these 8, he does not want to borrow Mathematics Part II, unless Mathematics Part I is also borrowed. In how many ways can he choose the 3 books to be borrowed?*

SOLUTION Clearly, the boy may borrow Mathematics Part II or he may not borrow it.

**Case I** *When the boy borrows Mathematics Part II:*

In this case, he borrows Mathematics Part I also.

Since he has 3 library tickets in all,

he may now borrow 1 book out of the remaining 6 books.

This can be done in ${}^6C_1 = 6$ ways.

**Case II** *When the boy does not borrow Mathematics Part II:*

Then, he may borrow 3 books out of the remaining 7 books.

This can be done in ${}^7C_3 = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$ ways.

Hence, the required number of ways $= (6 + 35) = 41$.

EXAMPLE 18 *A convex polygon has 44 diagonals. Find the number of its sides.*

SOLUTION Let the given convex polygon have $n$ sides.

Then, the number of its diagonals $= ({}^nC_2 - n)$.

$$\text{Now, } {}^nC_2 - n = 44 \Rightarrow \frac{n(n-1)}{2} - n = 44$$

$$\Rightarrow n^2 - 3n - 88 = 0 \Rightarrow (n-11)(n+8) = 0$$

$$\Rightarrow n = 11 \quad [\because n \neq -8].$$

Hence, the given polygon has 11 sides.

EXAMPLE 19 *If m parallel lines in a plane are intersected by a family of n parallel lines, find the number of parallelograms formed.*

SOLUTION A parallelogram is formed by choosing two straight lines from the set of $m$ parallel lines and two straight lines from the set of $n$ parallel lines.

Number of ways of choosing 2 lines out of $m$ parallel lines $= {}^mC_2$.

Number of ways of choosing 2 lines out of $n$ parallel lines $= {}^nC_2$.

Hence, the required number of parallelograms formed

$$= ({}^mC_2 \times {}^nC_2) = \frac{m(m-1)}{2} \times \frac{n(n-1)}{2} = \frac{mn(m-1)(n-1)}{4}.$$

EXAMPLE 20 *In an examination, a candidate has to pass in each of the 5 subjects. In how many ways can he fail?*

SOLUTION The candidate can fail by failing in 1 or 2 or 3 or 4 or 5 subjects out of 5 in each case.

$\therefore$ the total number of ways in which he can fail

$$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$$

$$= {}^5C_1 + {}^5C_2 + {}^5C_2 + {}^5C_1 + {}^5C_5 \qquad [\because {}^nC_r = {}^nC_{n-r}]$$

$$= (5 + 10 + 10 + 5 + 1) = 31.$$

EXAMPLE 21 *Determine the number of 5-card combinations out of a deck of 52 cards if there is exactly one ace in each combination.*

SOLUTION Number of ways of selecting 1 ace out of 4 aces $= {}^4C_1 = 4$.

Number of ways of selecting 4 cards out of remaining 48

$$= {}^{48}C_4 = \frac{48 \times 47 \times 46 \times 45}{4 \times 3 \times 2 \times 1} = 194580.$$

Hence, the required number of combinations

$$= (194580 \times 4) = 778320.$$

EXAMPLE 22 *In how many ways can one select a cricket team of eleven from 17 players in which only 5 players can bowl, if each cricket team of 11 must include exactly 4 bowlers?*

SOLUTION Number of ways of selecting 4 bowlers out of 5

$$= {}^5C_4 = {}^5C_{(5-4)} = {}^5C_1 = 5.$$

Number of ways of selecting 7 batsmen out of 12

$$= {}^{12}C_7 = {}^{12}C_{(12-7)} = {}^{12}C_5 = \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1} = 792.$$

Hence, the number of ways of selecting the team $= (5 \times 792) = 3960$.

EXAMPLE 23 *A committee of 12 persons is to be formed from 9 women and 8 men. In how many ways can this be done if at least 5 women have to be included in a committee? In how many of these committees are (a) the women in majority? (b) the men in majority?*

SOLUTION There are 9 women and 8 men in all. In order to form a committee consisting of at least 5 women, we may choose

(i) 5 women and 7 men or (ii) 6 women and 6 men

or (iii) 7 women and 5 men or (iv) 8 women and 4 men

or (v) 9 women and 3 men.

Hence, the total number of ways of forming the committee

$$= ({}^9C_5 \times {}^8C_7) + ({}^9C_6 \times {}^8C_6) + ({}^9C_7 \times {}^8C_5) + ({}^9C_8 \times {}^8C_4) + ({}^9C_9 \times {}^8C_3)$$

$$= ({}^9C_4 \times {}^8C_1) + ({}^9C_3 \times {}^8C_2) + ({}^9C_2 \times {}^8C_3) + ({}^9C_1 \times {}^8C_4) + (1 \times {}^8C_3)$$

$$= \left\{\frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times 8\right\} + \left\{\frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times \frac{8 \times 7}{2 \times 1}\right\} + \left\{\frac{9 \times 8}{2 \times 1} \times \frac{8 \times 7 \times 6}{3 \times 2 \times 1}\right\}$$

$$+ \left\{9 \times \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1}\right\} + \left(1 \times \frac{8 \times 7 \times 6}{3 \times 2 \times 1}\right)$$

$$= (1008 + 2352 + 2016 + 630 + 56) = 6062.$$

(a) Clearly, the women are in majority in (iii), (iv) and (v) cases.

So, the number of committees in which the women are in majority

$$= ({}^9C_7 \times {}^8C_5) + ({}^9C_8 \times {}^8C_4) + ({}^9C_9 \times {}^8C_3)$$

$$= ({}^9C_2 \times {}^8C_3) + ({}^9C_1 \times {}^8C_4) + (1 + {}^8C_3)$$

$$= (2016 + 630 + 56) = 2702.$$

(b) Clearly, the men are in majority in (i) only.

So, the number of committees in which the men are in majority

$$= ({}^9C_5 \times {}^8C_7) = ({}^9C_4 \times {}^8C_1) = \left(\frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times 8\right) = 1008.$$

EXAMPLE 24 *In a village, there are 87 families of which 52 families have at most 2 children. In a rural development programme, 20 families are to be chosen for assistance, of which at least 18 families must have at most 2 children. In how many ways can the choice be made?*

SOLUTION It is given that 52 families have at most 2 children and $(87 - 52) = 35$ families have more than 2 children.

We have to select 20 families of which at least 18 must have at most 2 children.

The selection can be made as under:

(i) 18 families out of 52 and 2 families out of 35

or (ii) 19 families out of 52 and 1 family out of 35

or (iii) 20 families out of 52.

So, the number of ways in which these choices can be made

$$= ({}^{52}C_{18} \times {}^{35}C_2) + ({}^{52}C_{19} \times {}^{35}C_1) + ({}^{52}C_{20}).$$

EXAMPLE 25 *In how many ways can 19 identical books on English and 17 identical books on Hindi be placed in a row on a shelf so that two books on Hindi may not be together?*

SOLUTION In order that two books on Hindi are never together, we must place these books as under:

$$XEXEXEX \ldots XEX,$$

where $E$ denotes the position of an English book and $X$ that of a Hindi book.

Since there are 19 books on English, the number of cross marks are therefore, 20.

Since all books on English are identical, they can be placed in only 1 way at the places marked *E*.

Number of ways of placing 17 identical books on Hindi at 20 places, each marked $X = {}^{20}C_{17} = {}^{20}C_3 = \dfrac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1140.$

Hence, the required number of ways $= (1 \times 1140) = 1140.$

## EXERCISE 9B

1. In how many ways can 5 sportsmen be selected from a group of 10?
2. A bag contains 5 black and 6 red balls. Find the number of ways in which 2 black and 3 red balls can be selected.
3. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 4 blue balls if each selection consists of 3 balls of each colour.
4. How many different boat parties of 8 consisting of 5 boys and 3 girls can be made from 20 boys and 10 girls?
5. In how many ways can a student choose 5 courses out of 9 courses if 2 specific courses are compulsory for every student?
6. A sports team of 11 students is to be constituted, choosing at least 5 from class XI and at least 5 from class XII. If there are 20 students in each of these classes, in how many ways can the team be constituted?
7. From 4 officers and 8 clerks, in how many ways can 6 be chosen (i) to include exactly one officer, (ii) to include at least one officer?
8. A cricket team of 11 players is to be selected from 16 players including 5 bowlers and 2 wicketkeepers. In how many ways can a team be selected so as to consist of exactly 3 bowlers and 1 wicketkeeper?
9. In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicketkeepers, assuming that the team of 11 players requires 5 batsmen, 3 all-rounders, 2 bowlers and 1 wicketkeeper?
10. A question paper has two parts, part A and part B, each containing 10 questions. If the student has to choose 8 from part A and 5 from part B, in how many ways can he choose the questions?
11. In an examination, a student has to answer 4 questions out of 5. Questions 1 and 2 are compulsory. Find the number of ways in which the student can make the choice.
12. In an examination, a student has to answer 10 questions, choosing at least 4 from each of part A and part B. If there are 6 questions in part A and 7 in part B, in how many ways can these questions be chosen?
13. In an examination, a candidate is required to answer 7 questions out of 12, which are divided into two groups, each containing 6 questions. One cannot attempt more than 5 questions from either group. In how many ways can he choose these questions?

**14.** Out of 6 teachers and 8 students, a committee of 11 is to be formed. In how many ways can this be done, if the committee contains

(i) exactly 4 teachers? (ii) at least 4 teachers?

**15.** A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of

(i) exactly 3 girls? (ii) at least 3 girls? (iii) at most 3 girls?

**16.** A committee of three persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?

**17.** A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways can this be done, when

(i) at least 2 ladies are included? (ii) at most 2 ladies are included?

**18.** From a class of 14 boys and 10 girls, 10 students are to be chosen for a competition, at least including 4 boys and 4 girls. The 2 girls who won the prizes last year should be included. In how many ways can the selection be made?

**19.** Find the number of 5-card combinations out of a deck of 52 cards if at least one of the five cards has to be a king.

**20.** Find the number of diagonals of

(i) a hexagon, (ii) a decagon, (iii) a polygon of 18 sides.

**21.** How many triangles can be obtained by joining 12 points, four of which are collinear?

**22.** How many triangles can be formed in a decagon?

**23.** How many different selections of 4 books can be made from 10 different books, if

(i) there is no restriction?

(ii) two particular books are always selected?

(iii) two particular books are never selected?

**24.** How many different products can be obtained by multiplying two or more of the numbers 3, 5, 7, 11 without repetition?

**25.** Find the number of ways in which a committee of 2 teachers and 3 students can be formed out of 10 teachers and 20 students. In how many of these committees

(i) a particular teacher is included?

(ii) a particular student is included?

(iii) a particular student is excluded?

**26.** There are 18 points in a plane of which 5 are collinear. How many straight lines can be formed by joining them?

***ANSWERS (EXERCISE 9B)***

**1.** 252 **2.** 200 **3.** 800 **4.** 1860480 **5.** 35 **6.** $2({}^{20}C_5 \times {}^{20}C_6)$

**7.** (i) 224 (ii) 896 **8.** 720 **9.** 141120 **10.** 11340 **11.** 3
**12.** 266 **13.** 780 **14.** (i) 120 (ii) 344 **15.** (i) 504 (ii) 588 (iii) 1632
**16.** 10, 6 **17.** (i) 186 (ii) 186 **18.** 266266 **19.** 886656
**20.** (i) 9 (ii) 35 (iii) 135 **21.** 216 **22.** 120 **23.** (i) 210 (ii) 28 (iii) 70
**24.** 11 **25.** 51300; (i) 10260 (ii) 7695 (iii) 43605 **26.** 144

## *HINTS TO SOME SELECTED QUESTIONS*

**5.** We have to choose 3 courses out of 7.

**6.** We have to choose:

(5 from XI and 6 from XII) or (6 from XI and 5 from XII)

Required number of ways $= \{(^{20}C_5 \times {}^{20}C_6) + (^{20}C_6 \times {}^{20}C_5)\}$.

**7.** (i) Number of ways of choosing (1 officer out of 4) and (5 clerks out of 8) in $(^4C_1 \times {}^8C_5) = (4 \times {}^8C_3)$ ways.

(ii) We may choose:

(1 officer, 5 clerks) or (2 officers, 4 clerks) or (3 officers, 3 clerks)
or (4 officers, 2 clerks).

Required number of ways

$= (^4C_1 \times {}^8C_5) + (^4C_2 \times {}^8C_4) + (^4C_3 \times {}^8C_3) + (^4C_4 \times {}^8C_2)$.

**8.** We have to choose 3 bowlers out of 5, 1 wicketkeeper out of 2 and 7 batsmen out of 9.

Required number of ways $= (^5C_3 \times {}^2C_1 \times {}^9C_7)$.

**9.** Required number of ways $= (^{10}C_5 \times {}^5C_3 \times {}^8C_2 \times {}^2C_1)$.

**10.** Required number of ways $= (^{10}C_8 \times {}^{10}C_5) = (^{10}C_2 \times {}^{10}C_5)$.

**11.** Number of ways of choosing 2 questions out of 3 $= {}^3C_2 = 3$.

**12.** He may choose:

(4 questions from A and 6 from B) or (5 from A and 5 from B)
or (6 from A and 4 from B).

Required number of ways $= (^6C_4 \times {}^7C_6) + (^6C_5 \times {}^7C_5) + (^6C_6 \times {}^7C_4)$
$= (^6C_2 \times {}^7C_1) + (^6C_1 \times {}^7C_2) + (1 \times {}^7C_3)$.

**13.** He may choose:

(5 from 1st group, 2 from 2nd group) or (4 from 1st group, 3 from 2nd group)
or (3 from 1st group, 4 from 2nd group) or (2 from 1st group, 5 from 2nd group).

Required number of ways

$= (^6C_5 \times {}^6C_2) + (^6C_4 \times {}^6C_3) + (^6C_3 \times {}^6C_4) + (^6C_2 \times {}^6C_5)$

$= (^6C_1 \times {}^6C_2) + (^6C_2 \times {}^6C_3) + (^6C_3 \times {}^6C_2) + (^6C_2 \times {}^6C_1)$.

**14.** (i) $(^6C_4 \times {}^8C_7) = (^6C_2 \times {}^8C_1)$.

(ii) $(^6C_4 \times {}^8C_7) + (^6C_5 \times {}^8C_6) + (^6C_6 \times {}^8C_5) = (^6C_2 \times {}^8C_1) + (^6C_1 \times {}^8C_2) + (1 \times {}^8C_3)$.

**15.** (i) $(^4C_3 \times {}^9C_4)$. (ii) $(^4C_3 \times {}^9C_4) + (^4C_4 \times {}^9C_3)$.

(iii) $(^9C_7) + (^9C_6 \times {}^4C_1) + (^9C_5 \times {}^4C_2) + (^9C_4 \times {}^4C_3)$
$= {}^9C_2 + (^9C_3 \times 4) + (^9C_4 \times {}^4C_2) + (^9C_4 \times {}^4C_1)$.

**16.** Total number of committees $= {}^5C_3 = {}^5C_2 = 10.$

Number of committees consisting of 1 man and 2 women

$= ({}^2C_1 \times {}^3C_2) = (2 \times {}^3C_1) = (2 \times 3) = 6.$

**17.** (i) $({}^4C_2 \times {}^6C_3) + ({}^4C_3 \times {}^6C_2) + ({}^4C_4 \times {}^6C_1).$

(ii) $({}^6C_5) + ({}^6C_4 \times {}^4C_1) + ({}^6C_3 \times {}^4C_2).$

**18.** 2 girls who won the prize last year are surely to be taken. So, we have to make a selection of 8 students out of 14 boys and 8 girls, choosing at least 4 boys and at least 2 girls.

Thus, we may choose:

(4 boys, 4 girls) or (5 boys, 3 girls) or (6 boys, 2 girls).

Required number of ways

$= ({}^{14}C_4 \times {}^8C_4) + ({}^{14}C_5 \times {}^8C_3) + ({}^{14}C_6 \times {}^8C_2).$

**19.** Required number of combinations

= (total number of 5-card combinations)

– (number of 5-card combinations having no king)

$= ({}^{52}C_5 - {}^{48}C_5).$

**20.** (i) Number of diagonals of a hexagon $= ({}^6C_2 - 6) = (15 - 6) = 9.$

(ii) Number of diagonals of a decagon $= ({}^{10}C_2 - 10) = (45 - 10) = 35.$

(iii) Number of diagonals of a polygon of 18 sides $= ({}^{18}C_2 - 18) = 135.$

**21.** Number of triangles $= ({}^{12}C_3 - {}^4C_3) = (220 - 4) = 216.$

**22.** Number of triangles in a decagon $= {}^{10}C_3 = 120.$

**23.** (i) Number of ways of selecting 4 books out of 10 $= {}^{10}C_4.$

(ii) After choosing 2 particular books, we have to choose 2 books out of remaining 8.

$\therefore$ required number of ways $= {}^8C_2.$

(iii) In this case, we have to choose 4 books out of remaining 8.

$\therefore$ required number of ways $= {}^8C_4.$

**24.** Required number of products

= number of ways of selecting 2 or 3 or 4 numbers out of 3, 5, 7, 11

$= ({}^4C_2 + {}^4C_3 + {}^4C_4) = (6 + 4 + 1) = 11.$

**25.** Number of ways of forming a committee of 2 teachers and 3 students $= ({}^{10}C_2 \times {}^{20}C_3).$

(i) Number of committees in which a particular teacher is included $= ({}^9C_1 \times {}^{20}C_3).$

(ii) Number of committees in which a particular student is included $= ({}^{10}C_2 \times {}^{19}C_2).$

(iii) Number of committees in which a particular student is excluded $= ({}^{10}C_2 \times {}^{19}C_3).$

**26.** Required number of straight lines $= ({}^{18}C_2 - {}^5C_2 + 1).$

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## MIXED PROBLEMS ON PERMUTATIONS AND COMBINATIONS

### SOLVED EXAMPLES

EXAMPLE 1 *Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?*

SOLUTION Total number of ways of choosing (3 consonants out of 7) and (2 vowels out of 4)

$$= ({}^7C_3 \times {}^4C_2) = \left(\frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1}\right) = 210.$$

Number of groups, each containing 3 consonants and 2 vowels = 210.

Each group contains 5 letters.

Number of ways of arranging 5 letters amongst themselves

$$= 5! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$$

Hence, the required number of words $= (210 \times 120) = 25200.$

EXAMPLE 2 *The English alphabet has 21 consonants and 5 vowels. How many words with two different consonants and two different vowels can be formed from the alphabet?*

SOLUTION Total number of ways of choosing

(2 consonants out of 21) and (2 vowels out of 5)

$$= {}^{21}C_2 \times {}^5C_2 = \left(\frac{21 \times 20}{2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = 2100.$$

Number of groups, each containing 2 consonants and 2 vowels = 2100.

Each group contains 4 letters.

Number of ways of arranging 4 letters amongst themselves

$$= 4! = (4 \times 3 \times 2 \times 1) = 24.$$

Hence, the required number of words $= (2100 \times 24) = 50400.$

EXAMPLE 3 *How many words with or without meaning, each of 2 vowels and 3 consonants can be formed from the letters of the word 'DAUGHTER'?*

SOLUTION The given word 'DAUGHTER' contains 3 vowels and 5 consonants.

Total number of ways of choosing

(2 vowels out of 3) and (3 consonants out of 5)

$$= ({}^3C_2 \times {}^5C_3) = ({}^3C_1 \times {}^5C_2) = \left(3 \times \frac{5 \times 4}{2 \times 1}\right) = 30.$$

Number of groups, each containing 2 vowels and 3 consonants = 30.

Each group contains 5 letters.

Number of ways of arranging 5 letters amongst themselves

$$= 5! = (5 \times 4 \times 3 \times 2 \times 1) = 120.$$

Hence, the required number of words $= (30 \times 120) = 3600.$

EXAMPLE 4 *How many words with or without meaning, can be formed using all the letters of the word 'EQUATION' at a time so that vowels and consonants occur together?*

SOLUTION The given word 'EQUATION' contains 5 vowels and 3 consonants.

All vowels can be put together in 5 ! ways.

All consonants can be put together in 3 ! ways.

Considering all vowels as 1 letter and all consonants as 1 letter, vowels and consonants can be arranged in 2 ! ways.

$\therefore$ required number of words

$$= (5!) \times (3!) \times (2!)$$
$$= (5 \times 4 \times 3 \times 2 \times 1) \times (3 \times 2 \times 1) \times (2 \times 1) = 1440.$$

<u>EXAMPLE 5</u> *In how many ways can 5 girls and 3 boys be seated in a row so that no two boys are together?*

SOLUTION Since no two boys are to be together, let us arrange 5 girls as under:

$$XGXGXGXGXGX,$$

where *G* denotes the position of a girl and *X* that of a boy.

These girls may be arranged at 5 places in 5 ! ways.

Now, 3 boys can be seated at 6 places, each marked *X*.

Number of ways of seating 3 boys at 6 places = $^6P_3$.

$\therefore$ required number of ways

$$= (5!) \times {}^6P_3 = (5!) \times \frac{6!}{3!}$$
$$= \left\{(5 \times 4 \times 3 \times 2 \times 1) \times \left(\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1}\right)\right\} = 14400.$$

Hence, the required number of ways is 14400.

## EXERCISE 9C

1. Out of 12 consonants and 5 vowels, how many words, each containing 3 consonants and 2 vowels, can be formed?
2. How many words, each of 3 vowels and 2 consonants, can be formed from the letters of the word 'INVOLUTE'?
3. The English alphabet has 21 consonants and 5 vowels. How many words with two different consonants and three different vowels can be formed from the alphabet?
4. In how many ways can 4 girls and 3 boys be seated in a row so that no two boys are together?
5. How many words, with or without meaning, can be formed from the letters of the word, 'MONDAY', assuming that no letter is repeated, if
   (i) 4 ketters are used at a time?
   (ii) all letters are used at a time?
   (iii) all letters are used but first letter is a vowel?

### ANSWERS (EXERCISE 9C)

**1.** 264000 **2.** 2880 **3.** 252000 **4.** 480

**5.** (i) 360 (ii) 720 (iii) 240

**5.** (i) Number of 4-letter words from the letters of 'MONDAY' = $^6P_4 = 360$.

(ii) Number of words formed from all the letters of 'MONDAY' = $^6P_6 = 6! = 720$.

(iii) The given word contains two vowels, namely *A* and *O*.

So, first letter of the word can be chosen in 2 ways.

Remaining 5 places can be filled in 5 ! ways.

$\therefore$ required number of words $= 2 \times (5!) = 240$.

## EXERCISE 9D

***Very-Short-Answer Questions***

**1.** If $^{20}C_r = {}^{20}C_{r-10}$ then find the value of $^{17}C_r$.

**2.** If $^{20}C_{r+1} = {}^{20}C_{r-1}$ then find the value of $^{10}C_r$.

**3.** If $^nC_{12} = {}^nC_8$ then find the value of $^{22}C_n$.

**4.** If $^{35}C_{n+7} = {}^{35}C_{4n-2}$ then find the value of *n*.

**5.** Find the values of (i) $^{200}C_{198}$, (ii) $^{76}C_0$, (iii) $^{15}C_{15}$.

**6.** If $^mC_1 = {}^nC_2$ prove that $m = \frac{1}{2}n(n-1)$.

**7.** Write the value of $({}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5)$.

**8.** If $^{n+1}C_3 = 2({}^nC_2)$, find the value of *n*.

**9.** If $^nP_r = 720$ and $^nC_r = 120$ then find the value of *r*.

**10.** If $^{(n^2-n)}C_2 = {}^{(n^2-n)}C_4$ then find the value of *n*.

**11.** How many words are formed by 2 vowels and 3 consonants, taken from 4 vowels and 5 consonants?

**12.** Find the number of diagonals in an *n*-sided polygon.

**13.** Three persons enter a railway compartment having 5 vacant seats. In how many ways can they seat themselves?

**14.** There are 12 points in a plane, out of which 3 points are collinear. How many straight lines can be drawn by joining any two of them?

**15.** In how many ways can a committee of 5 be made out of 6 men and 4 women, containing at least 2 women?

**16.** There are 13 cricket players, out of which 4 are bowlers. In how many ways can a team of 11 be selected from them so as to include at least 3 bowlers?

**17.** How many different committees of 5 can be formed from 6 men and 4 women, if each committee consists of 3 men and 2 women?

**18.** How many parallelograms can be formed from a set of 4 parallel lines intersecting another set of 3 parallel lines?

## ANSWERS (EXERCISE 9D)

| | | | | | |
|---|---|---|---|---|---|
| **1.** 136 | **2.** 1 | **3.** 231 | **4.** $n=6$ | **5.** (i) 19900 (ii) 1 (iii) 1 | |
| **7.** 31 | **8.** $n=5$ | **9.** $r=3$ | **10.** $n=3$ | **11.** 60 | **12.** $\frac{n(n-3)}{2}$ |
| **13.** 60 | **14.** 64 | **15.** 186 | **16.** 72 | **17.** 120 | **18.** 12 |

## HINTS TO SOME SELECTED QUESTIONS

**1.** $r+r-10=20 \Rightarrow r=15$. So, ${}^{17}C_r = {}^{17}C_{15} = {}^{17}C_2 = \frac{17\times 16}{2\times 1} = 136.$

**2.** $r+1+r-1=20 \Rightarrow r=10$. So, ${}^{10}C_r = {}^{10}C_{10} = 1.$

**3.** $n=12+8=20 \Rightarrow {}^{22}C_n = {}^{22}C_{20} = {}^{22}C_2 = \frac{22\times 21}{2} = 231.$

**4.** $n+7+4n-2=35 \Rightarrow 5n=30 \Rightarrow n=6.$

**5.** (i) ${}^{200}C_{198} = {}^{200}C_2 = \frac{200\times 199}{2} = 19900.$

(ii) ${}^{76}C_0 = 1.$

(iii) ${}^{15}C_{15} = 1.$

**6.** ${}^{m}C_1 = {}^{n}C_2 \Rightarrow m = \frac{n(n-1)}{2}.$

**7.** ${}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = ({}^5C_1 + {}^5C_2 + {}^5C_2 + {}^5C_1 + {}^5C_5) = (5+10+10+5+1) = 31.$

**8.** ${}^{n+1}C_3 = 2({}^nC_2) \Rightarrow \frac{(n+1)n(n-1)}{6} = \frac{n(n-1)}{2}\times 2 \Rightarrow n+1=6 \Rightarrow n=5.$

**9.** ${}^nP_r = ({}^nC_r)\times(r!) \Rightarrow r! = \frac{720}{120} = 6 = 3\times 2\times 1 = 3! \Rightarrow r=3.$

**10.** $n^2-n=(2+4) \Rightarrow n^2-n-6=0 \Rightarrow (n-3)(n+2)=0 \Rightarrow n=3.$

**11.** Number of words $= ({}^4C_2 \times {}^5C_3) = ({}^4C_2 \times {}^5C_2) = (6\times 10) = 60.$

**12.** Required number of diagonals $= ({}^nC_2 - n) = \frac{n(n-1)}{2} - n = \frac{n^2-n-2n}{2}$

$$= \frac{n^2-3n}{2} = \frac{n(n-3)}{2}.$$

**13.** First can occupy any of the 5 seats, 2nd may occupy any of 4 seats and third may occupy any of 3 seats.

Required number of ways $= (5\times 4\times 3) = 60.$

**14.** Required number of straight lines $= ({}^{12}C_2 - {}^3C_2 + 1) = 64.$

**15.** (2 women and 3 men) or (3 women and 2 men) or (4 women and 1 man)

$= ({}^4C_2 \times {}^6C_3) + ({}^4C_3 \times {}^6C_2) + ({}^4C_4 \times {}^6C_1) = (120+60+6) = 186.$

**16.** There are 4 bowlers and 9 batsmen.

We may choose (3 bowlers and 8 batsmen) or (4 bowlers and 7 batsmen).

Required number of ways $= ({}^4C_3 \times {}^9C_8) + ({}^4C_4 \times {}^9C_7) = 36+36 = 72.$

**17.** Number of committees $= ({}^6C_3 \times {}^4C_2) = (20\times 6) = 120.$

**18.** Number of parallelograms $= (4\times 3) = 12.$